अध्याय

8 ग्रेडिएंट, डाइवर्जेन्स तथा कर्ल (Gradient, Divergence and Curl)

8.1 परिभाषायें (Definitions)

(A) 1. डेल ऑपरेटर या सदिश अवकल ऑपरेटर (Del Operator or Vector Operator)

सदिश संकारक ∇, जिसे डेल (Del) या नैब्ला (Nabla) पढ़ा जाता है, को निम्न रूप में परिभाषित किया जाता है :

$$\nabla \equiv \hat{\mathbf{i}}\frac{\partial}{\partial x}+\hat{\mathbf{j}}\frac{\partial}{\partial y}+\hat{\mathbf{k}}\frac{\partial}{\partial z}$$

2. अदिश बिन्दु फलन का ग्रेडिएंट (प्रवणता) (Gradient of a Scalar Point Function)

यदि $\phi(x, y, z)$ एक संगत तथा अवकलनीय अदिश बिन्दु फलन है तो सदिश फलन

$$\nabla\phi=\left(\hat{\mathbf{i}}\frac{\partial}{\partial x}+\hat{\mathbf{j}}\frac{\partial}{\partial y}+\hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\phi=\hat{\mathbf{i}}\frac{\partial\phi}{\partial x}+\hat{\mathbf{j}}\frac{\partial\phi}{\partial y}+\hat{\mathbf{k}}\frac{\partial\phi}{\partial z}$$

को ϕ का ग्रेडिएंट (gradient) कहते हैं तथा इसे grad ϕ या $\nabla\phi$ से सूचित किया जाता है।

(B) 1. सदिश फलन का डाइवर्जेन्स (Divergence of a Vector Function)

यदि $\vec{\mathbf{F}}(x, y, z)$ कोई संतत अवकलनीय सदिश फलन हो तो डेल ऑपरेटर ∇ तथा सदिश फलन $\vec{\mathbf{F}}$ के अदिश गुणन $\nabla.\vec{\mathbf{F}}$ को $\vec{\mathbf{F}}$ का डाइवर्जेन्स (Divergence) कहते हैं तथा इसे div $\vec{\mathbf{F}}$ से व्यक्त किया जाता है।

अतः div $\vec{\mathbf{F}}=\nabla.\vec{\mathbf{F}}=\left(\hat{\mathbf{i}}\frac{\partial}{\partial x}+\hat{\mathbf{j}}\frac{\partial}{\partial y}+\hat{\mathbf{k}}\frac{\partial}{\partial z}\right).\vec{\mathbf{F}}=\hat{\mathbf{i}}.\frac{\partial\vec{\mathbf{F}}}{\partial x}+\hat{\mathbf{j}}.\frac{\partial\vec{\mathbf{F}}}{\partial y}+\hat{\mathbf{k}}.\frac{\partial\vec{\mathbf{F}}}{\partial z}$

$$=\frac{\partial Fx}{\partial x}+\frac{\partial Fy}{\partial y}+\frac{\partial Fz}{\partial z}$$

जहाँ $\vec{\mathbf{F}}=F_x\,\hat{\mathbf{i}}+F_y\,\hat{\mathbf{j}}+F_z\,\hat{\mathbf{k}}$, div एक अदिश राशि है।

2. सालेनॉयडल सदिश (Solenoidal Vector)

यदि $\vec{\mathbf{V}}$ कोई सदिश हो तथा div $\vec{\mathbf{V}}=0$ तो $\vec{\mathbf{V}}$ को सोलेनॉयडल सदिश कहते हैं।

C. 1. किसी सदिश का कर्ल (Curl of a Vector)

यदि $\vec{\mathbf{F}}$ कोई सतत अवकलनीय (continuous differentiable) सदिश फलन हो तो डेल ऑपरेटर ∇ तथा $\vec{\mathbf{F}}$ का सदिश गुणन $\nabla\times\vec{\mathbf{F}}$ फलन $\vec{\mathbf{F}}$ का कर्ल (Curl) या रोटेशन (Rotation) कहलाता है तथा इसे curl $\vec{\mathbf{F}}$ या rot $\vec{\mathbf{F}}$ से सूचित करते हैं। अतः यदि $\vec{\mathbf{F}}=F_1\hat{\mathbf{i}}+F_2\hat{\mathbf{j}}+F_3\hat{\mathbf{k}}$

$$\text{curl }\vec{\mathbf{F}}=\vec{\nabla}\times\vec{\mathbf{F}}=\left(\hat{\mathbf{i}}\frac{\partial}{\partial x}+\hat{\mathbf{j}}\frac{\partial}{\partial y}+\hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\times\vec{\mathbf{F}}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix}$$

किसी सदिश फलन $\vec{\mathbf{F}}$ का कर्ल एक सदिश राशि है। जहाँ $\vec{\mathbf{F}} = F_1\hat{\mathbf{i}} + F_2\hat{\mathbf{j}} + F_3\hat{\mathbf{k}}$

2. इरोटेशलन सदिश (Irrotational Vector)

कोई सदिश $\vec{\mathbf{F}}$ इरोटेशलन कहलाता है यदि $\nabla \times \vec{\mathbf{F}} = 0$ अर्थात् उसका कर्ल शून्य हो।

8.2 प्रवणता पर आधारित महत्वपूर्ण परिणाम (Important Results Involving Gradients)

(i) यदि f कोई अचर फलन हो, तो $\nabla \boldsymbol{f} = \mathbf{0}$

(ii) यदि f तथा g दो अदिश बिन्दु फलन हैं तो grad $(f \pm g)$ = grad $f \pm$ grad g
या $\nabla (f \pm g) = \nabla f \pm \nabla g$

(iii) grad $(fg) = f$ grad $g + g$ grad f अर्थात् $\nabla (fg) = f\,\nabla g + g\,\nabla f$

(iv) $\text{grad}\left(\frac{f}{g}\right) = \frac{g \text{ grad } f - f \text{ grad } g}{g^2}$ या $\nabla\left(\frac{f}{g}\right) = \frac{g\,\nabla f - f\,\nabla g}{g^2}$

(v) यदि कोई अचर सदिश हो तो div $\vec{\mathbf{A}} = 0$

(vi) यदि $\vec{\mathbf{A}}$ तथा $\vec{\mathbf{B}}$ दो सदिश फलन हैं, तो div $(\vec{\mathbf{A}} \pm \vec{\mathbf{B}})$ = div $\vec{\mathbf{A}} \pm$ div $\vec{\mathbf{B}}$

(vii) यदि $\vec{\mathbf{A}}$ अचर सदिश है, तो curl $\vec{\mathbf{A}} = 0$ [200

(viii) यदि $\vec{\mathbf{A}}$ तथा $\vec{\mathbf{B}}$ दो सदिश फलन हों तो

(ix) curl $(\vec{\mathbf{A}} \pm \vec{\mathbf{B}})$ = curl $\vec{\mathbf{A}} \pm$ curl $\vec{\mathbf{B}}$ या $\nabla \times (\vec{\mathbf{A}} \pm \vec{\mathbf{B}}) = \nabla \times \vec{\mathbf{A}} \pm \nabla \times \vec{\mathbf{B}}$

परीक्षा में पूछे गए एवं अन्य महत्वपूर्ण प्रश्न

वस्तुनिष्ठ प्रश्न

1. डेल आपॅरेटर $\nabla =$

(a) $\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ (b) $-\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)$

(c) $\mathbf{i}\frac{\partial}{\partial x} - \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial}$ (d) कोई नहीं

2. यदिइ ϕ एक अदिश फलन हो, तो $\nabla\phi =$

(a) $\mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}$ (b) $-\left(\mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}\right)$

(c) $\left(\mathbf{i}\frac{\partial}{\partial x} - \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)$ (d) कोई नहीं

3. यदि $\mathbf{r} = x\mathbf{i} + y\,\mathbf{j} + z\mathbf{k}$ तो grad $r =$

(a) r (b) $\hat{\mathbf{r}}$ (c) 0 (d) कोई नहीं

4. यदि $|\vec{\mathbf{r}}| = r$ तथा $\vec{\mathbf{r}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ तो grad $r =$

(a) r (b) $\mathbf{r}$ (c) $\mathbf{i} + \mathbf{j} + \mathbf{k}$ (d) कोई नहीं

5. यदि $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ तो $div\,\mathbf{r} =$

(a) 1 (b) 2 (c) 3 (d) कोई नहीं

6. यदि ϕ के अदिश फलन है तो curl grad $\phi =$

(a) 0 (b) 1 (c) 2 (d) कोई नहीं

7. यदि $\mathbf{F}$ कोई सदिश हो तो Curl $\mathbf{F} =$

(a) ∇F (b) $\nabla \times F$ (c) $\nabla . F$ (d) कोई नहीं

8. यदि $F = F_1 i + F_2 j + F_3 k$ तो $\nabla \times F =$

(a) $\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$ (b) $\begin{vmatrix} i & j & k \\ F_1 & F_2 & F_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$ (c) $\begin{vmatrix} F_1 & F_2 & F_3 \\ i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$ (d) कोई नहीं

9. यदि F irrotational हो तो Curl F =

(a) 0 (b) 1 (c) 2 (d) कोई नहीं

✍ अति लघुउत्तरीय प्रश्न

1. (a) यदि $\vec{F} = x^2 y\hat{i} - 2xz\hat{j} + 2yz\hat{k}$, तो $\vec{\nabla} . \vec{F}$ का मान ज्ञात कीजिए। **[2022(S)]**
   (b) यदि $\bar{\phi}(x, y, z) = 3x^2 + 2y^2 + z^2$, grad $\bar{\phi}$ ज्ञात कीजिए। **[2021(S)]**
   (c) दिखाइये कि $(a.\nabla) r = a$ **[2019(S)]**
   (d) यदि $\vec{F} = xy^3 - x^3 y$ तो $\nabla^2 F$ का मान ज्ञात करें। **[2016]**
2. यदि $\phi(x, y, z) = 3x^2 y - y^3 z^2$ तो $\nabla\phi$ का मान बिन्दु $(1, -2, -1)$ पर ज्ञात करें। **[2015]**
3. यदि $f(x, y, z) = x^3 + y^3 + z^3 + 3xyz$ तो ∇f ज्ञात करें। **[2010, 14]**
4. सिद्ध करें div $\vec{r} = 3$, जहाँ $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ **[2010, 06]**
5. $x^2 y + 2xy - 4$ की प्रवणता बिंदु $(2, -2, 3)$ पर ज्ञात करें।
6. $\phi = x^3 - y^3 + xz^2$ के बिन्दु $(1, -1, 2)$ पर grad ϕ ज्ञात करें। **[2004]**

✍ लघुउत्तरीय प्रश्न/दीर्घउत्तरीय प्रश्न

1. (a) यदि $\phi = \log(x^2 + y^2 + z^2)$ तो $\vec{\nabla}\phi$ का मान ज्ञात कीजिए, जहाँ $r^2 = x^2 + y^2 + z^2$ और $\vec{r} = x\hat{i} + y\hat{j} = z\hat{k}$ **[2022(S)]**
   (b) दर्शाइये कि $(\vec{a} . \nabla) \vec{r} = \vec{a}$ **[2019(S)]**
   (c) सदिश फलन $\vec{f} = (x^2 - y^2)\hat{i} + 2xy\,\hat{j} + (y^2 - 2xy)\hat{k}$ का डाइवर्जेन्स और कर्ल ज्ञात कीजिए। **[2003, 18(SB)]**
   (d) सदिश फलन $\vec{f} = (x^2 - y^2)\hat{i} + 2xy\,\hat{j} + (y^2 - xy)\hat{k}$ का डाइवर्जेन्स और कर्ल ज्ञात कीजिए। **[2021(S)]**
2. (i) यदि $\vec{V} = x^2 z\,\hat{i} - 2y^3 z^2\,\hat{j} + xy^2 z\,\hat{k}$ तो curl $\vec{V}$ का मान बिंदु $(1, -1, 1)$ पर ज्ञात करें। **[2015]**
   (ii) यदि $\vec{F} = xy^2\,\hat{i} + 2x^2 yz\,\hat{j} - 3yz^2\,\hat{k}$ तो $\nabla . \vec{F}$ का मान बिंदु $(1, -1, 1)$ पर ज्ञात करें। **[2015]**
3. div $\vec{F}$ का मान ज्ञात करें, जहाँ $\vec{F} = \text{grad}\,(x^3 + y^2 + z^3 - 3xyz)$ **[2014]**
4. (i) यदि $r^2 = x^2 + y^2 + z^2$ तो grad $\bar{r}^n$ का मान ज्ञात करें। **[2013]**
   (ii) यदि $r = |\vec{r}|$ जहाँ $r = x\hat{i} + y\hat{j} + z\hat{k}$, तो सिद्ध करो $\nabla r = \frac{1}{r}\vec{r}$. **[2019(S)]**
5. बिन्दु $(2, -1, 2)$ पर z-अक्ष के साथ $4xz^3 - 3x^2 y^2 z^2$ का डाइरेक्शनल डेरीवेटिव (directional derivative) ज्ञात कीजिए। **[2007, 12]**

6. $(x^2+y^2+z^2)$ का बिन्दु (1, 2, 3) पर रेखा $\frac{x}{2}=\frac{y}{2}=\frac{z}{1}$ की दिशा में दिशात्मक व्युत्पत्ति दिए अवकल (directional derivative) ज्ञात करें। [2011]

7. (i) यदि $\vec{\mathbf{F}}=(x+y+1)\hat{\mathbf{i}}+\hat{\mathbf{j}}-(x+y)\hat{\mathbf{k}}$ तो सिद्ध करें $\vec{\mathbf{F}}.\text{curl}\,\vec{\mathbf{F}}=0$ [2006, 08, 10

(ii) यदि $\phi=\frac{1}{r}$ जहाँ $\vec{\mathbf{r}}=x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}$ तथा $r^2=x^2+y^2+z^2$ तो $\nabla\phi$ का मान प्राप्त करें। [2010

हल एवं संकेत

वस्तुनिष्ठ प्रश्न

1. (a) 2. (a) 3. (b) 4. (d) 5. (c) 6. (a) 7. (b) 8. (a)
9. (b)

अति लघुउत्तरीय प्रश्न

1. (a) $\because\ \vec{\nabla}.\vec{F}=\frac{\partial}{\partial x}F_x+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z},$

यहाँ $F_x=x^2y.F_y=-2xz, F_z=2yz$

$$\because\quad \vec{\nabla}.\vec{F}=\frac{\partial}{\partial x}(x^2y)+\frac{\partial}{\partial y}(-2xz)+\frac{\partial}{\partial z}(2yz)=2xy-0+2y=2y\,(x+1)$$

(b) grad $\phi=\nabla\,(3x^2+2y^2+x^2)$

$$=\left(\hat{\mathbf{i}}\frac{\partial}{\partial x}+\hat{\mathbf{j}}\frac{\partial}{\partial y}+\hat{\mathbf{k}}\frac{\partial}{\partial z}\right)(3x^2+3y^2+3z^2)=\hat{\mathbf{i}}\frac{\partial}{\partial x}(3x^2)+\hat{\mathbf{j}}\frac{\partial}{\partial y}(3y^2)+\hat{\mathbf{k}}\frac{\partial}{\partial z}(3z^2)$$

$$=2\times 2x\hat{\mathbf{i}}+3\times 2y\hat{\mathbf{j}}+3\times 2z\hat{\mathbf{k}}=6(x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}})$$

(c) माना $\vec{a}=a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}}$ तथा $\vec{\mathbf{r}}=x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}$

$$\therefore\quad (\vec{a}.\nabla)\vec{\mathbf{r}}=\left\{(a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}}).\left(\hat{\mathbf{i}}\frac{\partial}{\partial x}+\hat{\mathbf{j}}\frac{\partial}{\partial y}+\hat{\mathbf{k}}\frac{d}{\partial z}\right)\right\}\vec{\mathbf{r}}$$

$$=\left(a_1\frac{\partial}{\partial x}+a_2\frac{\partial}{\partial y}+a_3\frac{\partial}{\partial z}\right).(x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}})=a_1\hat{\mathbf{i}}\frac{\partial x}{\partial x}+a_2\hat{\mathbf{j}}\frac{\partial y}{\partial y}+a_3\hat{\mathbf{k}}\frac{\partial z}{\partial z}$$

$$=a_1\hat{\mathbf{i}}\times 1+a_2\hat{\mathbf{j}}\times 1+a_3\hat{\mathbf{k}}\times 1=a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}}=\vec{a}$$

(d) $\mathbf{F}=xy^3-x^3y$

$$\therefore\quad \nabla\mathbf{F}=\left(\hat{\mathbf{i}}\frac{\partial}{\partial x}+\hat{\mathbf{j}}\frac{\partial}{\partial y}+\hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\mathbf{F}=\hat{\mathbf{i}}\frac{\partial F}{\partial x}+\hat{\mathbf{j}}\frac{\partial F}{\partial y}+\hat{\mathbf{k}}\frac{\partial F}{\partial z}$$

$$=\hat{\mathbf{i}}\frac{\partial}{\partial x}(xy^3-x^3y)+\hat{\mathbf{j}}\frac{\partial}{\partial y}(xy^3-x^3y)+\hat{\mathbf{k}}\frac{\partial}{\partial z}(xy^3-x^3y)$$

$$=\hat{\mathbf{i}}(y^3-3x^2y)+\hat{\mathbf{j}}(3xy^2-3x^2)+0.\hat{\mathbf{k}}$$

$$\therefore\quad \nabla^2\mathbf{F}=\nabla.\nabla\mathbf{F}=\left(\hat{\mathbf{i}}\frac{\partial}{\partial x}+\hat{\mathbf{j}}\frac{\partial}{\partial y}+\hat{\mathbf{k}}\frac{\partial}{\partial z}\right).\{\hat{\mathbf{i}}(y^3-3x^2y)+\hat{\mathbf{j}}(3xy^2-3x^2)\}+0.\hat{\mathbf{k}}$$

$$=\hat{\mathbf{i}}.\hat{\mathbf{i}}\frac{\partial}{\partial x}(y^3-3x^2y)+\hat{\mathbf{j}}.\hat{\mathbf{j}}\frac{\partial}{\partial y}(3xy^2-3x^2)+0=-1\times 6xy+1\times 6xy=0$$

2. $\phi(x,y,z)=3x^2y-y^3z^2$

$\therefore \quad \nabla\phi = \hat{\mathbf{i}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{j}}\frac{\partial\phi}{\partial y} + \hat{\mathbf{k}}\frac{\partial\phi}{\partial z}$

$= \hat{\mathbf{i}}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \hat{\mathbf{j}}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \hat{\mathbf{k}}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$

$= \hat{\mathbf{i}}(6xy) + \hat{\mathbf{j}}(3x^2 - 3y^2z^2) + \hat{\mathbf{k}}(-2y^3z)$

$\therefore$ बिन्दु $(1, -2, -1)$ पर

$(\nabla\phi)(1, -2, -1) = \hat{\mathbf{i}}\{6\times 1\times(-2)\} + \hat{\mathbf{j}}\{3\times 1^2 - 3\times(-2)^2\times(-1)^2 \;\; \hat{\mathbf{k}}\{-2\times(-2)^3\times(-1)\}$

$= -12\hat{\mathbf{i}} - 9\hat{\mathbf{j}} - 16\hat{\mathbf{k}} = -(12\hat{\mathbf{i}} + 9\hat{\mathbf{j}} + 16\hat{\mathbf{k}})$

3. $f(x, y, z) = x^3 + y^3 + z^3 + 3xyz$

$\therefore \quad \nabla f = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)(x^3 + y^2 + z^3 + 3xyz)$

$= \hat{\mathbf{i}}\frac{\partial}{\partial x}(x^3 + y^3 + z^3 + 3xyz) + \hat{\mathbf{j}}\frac{\partial}{\partial y}(x^3 + y^3 + z^3 + 3xyz) + \hat{\mathbf{k}}\frac{\partial}{\partial z}(x^3 + y^2 + z^3 + 3xyz)$

$= \hat{\mathbf{i}}(3x^2 + 3yz) + \hat{\mathbf{j}}(3y^2 + 3xz) + \hat{\mathbf{k}}(3z^2 + 3xy)$

$= 3[(x^2 + yz)\hat{\mathbf{i}} + (y^2 + xz)\hat{\mathbf{j}} + (z^2 + xy)\hat{\mathbf{k}}]$

4. प्रश्न से $\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \quad \Rightarrow \quad \frac{\partial\vec{\mathbf{r}}}{\partial x} = \hat{\mathbf{i}}, \frac{\partial\vec{\mathbf{r}}}{\partial y} = \hat{\mathbf{j}}, \frac{\partial\vec{\mathbf{r}}}{\partial z} = \hat{\mathbf{k}}$

$\therefore$ परिभाषा से $\text{div}\,\vec{\mathbf{r}} = \nabla.\vec{\mathbf{r}} = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}} + \frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right).\vec{\mathbf{r}} = \hat{\mathbf{i}}.\frac{\partial\vec{\mathbf{r}}}{dx} + \hat{\mathbf{j}}.\frac{\partial\vec{\mathbf{r}}.}{\partial y} + \hat{\mathbf{k}}.\frac{\partial\vec{\mathbf{r}}}{\partial z}$

$= \hat{\mathbf{i}}.\hat{\mathbf{i}} + \hat{\mathbf{j}}.\hat{\mathbf{j}} + \hat{\mathbf{k}}.\hat{\mathbf{k}} = 1 + 1 + 1 = 3 \qquad [\because \hat{\mathbf{i}}.\hat{\mathbf{i}} = \hat{\mathbf{j}}.\hat{\mathbf{j}} = \hat{\mathbf{k}}.\hat{\mathbf{k}} = 1]$

5. माना $f = x^2y + 2xz - 4$ तो grad f

$\text{grad}\,(x^2y + 2xz - 4) = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right).(x^2y + 2xz - 4)$

$= (2xy + 2z)\hat{\mathbf{i}} + x^2\hat{\mathbf{j}} + 2x\hat{\mathbf{k}}) \qquad \ldots(1)$

अतः बिंदु $(2, -2, 3)$ पर grad $f = [2\times 2\times(-2) + 2\times 3]\hat{\mathbf{i}} + 2^2\hat{\mathbf{j}} + 2\times 2\hat{\mathbf{k}} = -2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$

6. प्रश्न से, $f = x^3 - y^3 + xz^2$

$\therefore \quad \text{grad}\,f = \nabla f = (\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z})(x^3 - y^3 + xz^2)$

$= \hat{\mathbf{i}}\left\{\frac{\partial}{\partial x}(x^3 - y^3 + xz^2)\right\} + \hat{\mathbf{j}}\left\{\frac{\partial}{\partial y}(x^3 - y^3 + xz^2)\right\} + \hat{\mathbf{k}}\left\{\frac{\partial}{\partial z}(x^3 - y^3 + xz^2)\right\}$

$= \hat{\mathbf{i}}\{3x^2 - 0 + z^2\} + \hat{\mathbf{j}}\{0 - 3y^2 + 0\} + \hat{\mathbf{k}}\{0 - 0 + 2xz\}$

अर्थात् $\nabla f = (3x^2 + z^2)\hat{\mathbf{i}} - 3y^2\hat{\mathbf{j}} + 2xz\hat{\mathbf{k}} \qquad \ldots(1)$

$\therefore$ बिन्दु $(1, -1, 2)$, $(\nabla f)_{(1,-2,2)} = (3\times 1^2 + 2^2)\hat{\mathbf{i}} - 3\times(-1)^2\hat{\mathbf{j}} + 2\times 1\times 2\hat{\mathbf{k}}$

[(1) में $x = 1, y = -1$ तथा $z = 2$ रखने पर]

$= 7\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ उत्तर

लघुउत्तरीय प्रश्न/दीर्घउत्तरीय प्रश्न

1. (a) $\nabla\phi = \nabla\{\log(x^2 + y^2 + z^2) + \hat{\mathbf{i}}\frac{\partial}{\partial x}\log(x^2 + y^2 + z^2) + \hat{\mathbf{j}}\frac{\partial}{\partial y}\log(x^2 + y^2 + z^2)$

$= \hat{\mathbf{i}}\frac{1}{x^2 + y^2 + z^2}\times 2x + \hat{\mathbf{j}}\frac{1}{x^2 + y^2 + z^2}\times 2y + \hat{\mathbf{k}}\frac{1}{x^2 + y^2 + z^2}\times 2z$

$$= 2\left[\hat{\mathbf{i}}\frac{x}{x^2+y^2+z^2} + \hat{\mathbf{j}}\frac{y}{x^2+y^2+z^2} + \hat{\mathbf{k}}\frac{z}{x^2+y^2+z^2}\right]$$

$$= 2\left[\frac{x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}}{x^2+y^2+z^2}\right] = \frac{2\vec{\mathbf{r}}}{x^2+y^2+z^2} = \frac{2\vec{\mathbf{r}}}{r^2}$$

$[\because r^2 = x^2+y^2+z^2 \Rightarrow \vec{\mathbf{r}} = x\hat{\mathbf{i}} - y\hat{\mathbf{j}} +$

(b) Let $\vec{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

$$\therefore \quad (\vec{a}\,.\nabla)\,\vec{\mathbf{r}} = \left\{(a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}}).\left(\hat{\mathbf{i}}\frac{\partial}{\partial x}+\hat{\mathbf{j}}\frac{\partial}{\partial y}+\hat{\mathbf{k}}\frac{d}{\partial z}\right)\right\}\vec{\mathbf{r}}$$

$$= \left(a_1\frac{\partial}{\partial x}+a_2\frac{\partial}{\partial y}+a_3\frac{\partial}{\partial z}\right).(x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}) = a_1\hat{\mathbf{i}}\frac{\partial x}{\partial x}+a_2\hat{\mathbf{j}}\frac{\partial y}{\partial y}+a_3\hat{\mathbf{k}}\frac{\partial z}{\partial z}$$

$$= a_1\hat{\mathbf{i}}\times 1 + a_2\hat{\mathbf{j}}\times 1 + a_3\hat{\mathbf{k}}\times 1 = a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}} = \vec{a}$$

(c) प्रश्न से, $\vec{\mathbf{f}} = (x^2-y^2)\,\hat{\mathbf{i}} + 2xy\,\hat{\mathbf{j}} + (y^2-2xy)\,\hat{\mathbf{k}}$

$$\therefore \quad \text{div}\,(\vec{\mathbf{f}}) = \nabla\,.\,\vec{\mathbf{f}} = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x}+\hat{\mathbf{j}}\frac{\partial}{\partial y}+\hat{\mathbf{k}}\frac{\partial}{\partial z}\right).[(x^2-y^2)\,\hat{\mathbf{i}}+2xy\hat{\mathbf{j}}+(y^2-2xy)\,\hat{\mathbf{k}}]$$

$$= \frac{\partial}{\partial x}(x^2-y^2)+\frac{\partial}{\partial y}(2xy)+\frac{\partial}{\partial z}(y^2-2xy) = 2x+2x+0 = 4x$$ उत्तर

पुनः $$\text{curl}\,\vec{\mathbf{f}} = \nabla\times\vec{\mathbf{f}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^2 & 2xy & y^2-2xy \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(y^2-2xy)-\frac{\partial}{\partial z}(2xy)\right]\hat{\mathbf{i}} - \left[\frac{\partial}{\partial x}(y^2-2xy)-\frac{\partial}{\partial z}(x^2-y^2)\right]\hat{\mathbf{j}}$$

$$+\left[\frac{\partial}{\partial x}(2xy)-\frac{\partial}{\partial y}(x^2-y^{2)}\right]$$

$$= [2y-2x-0]\,\hat{\mathbf{i}} - [-2y-0]\,\hat{\mathbf{j}} + (2y+2y)\,\hat{\mathbf{k}}$$

$$= 2(y-x)\,\hat{\mathbf{i}} + 2y\,\hat{\mathbf{j}} + 4y\,\hat{\mathbf{k}}$$

उ

(d) (c) की तरह हल करें।

2. (i) $V = x^2z\hat{\mathbf{i}} - 2y^3z^2\hat{\mathbf{j}} + xy^2z\hat{\mathbf{k}}$

$$\therefore \quad \text{Curl } V = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z & -2y^3z^2 & xy^2z \end{vmatrix}$$

$$= \hat{\mathbf{i}}\left[\frac{\partial}{\partial y}(xy^2z)-\frac{\partial}{\partial z}(-2y^3z^2)\right] - \hat{\mathbf{j}}\left[\frac{\partial}{\partial x}(xy^2z)-\frac{\partial}{\partial z}(x^2z)\right]$$

$$+\hat{\mathbf{k}}\left[\frac{\partial}{\partial x}(-xy^3z^2)-\frac{\partial}{\partial y}(x^2z)\right]$$

$$= \hat{\mathbf{i}}\,(2xyz+4y^3z) - \hat{\mathbf{j}}\,(y^2z-x^2) + \hat{\mathbf{k}}\,(0-0)$$

$\therefore$ बिन्दु (1, −1, 1) पर curl $V = \hat{\mathbf{i}}(2\times1\times(-1)\times1+4(-1)^3\times1] - \hat{\mathbf{j}}[(-1)^2\times1-1^2] = -6\hat{\mathbf{i}}$

(ii) $\vec{\mathbf{F}} = xy^2\,\hat{\mathbf{i}} + 2x^2yz\,\hat{\mathbf{j}} - 3yz^2\hat{\mathbf{k}}$

$\therefore$ div $\vec{\mathbf{F}} = \nabla\,.\,\vec{\mathbf{F}}\left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right).(xy^2\,\hat{\mathbf{i}} + 2xy^2z\,\hat{\mathbf{j}} - 3yz^2\,\hat{\mathbf{k}})$

$$= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) - \frac{\partial}{\partial z}(3yz^2) = y^2 + 2x^2z - 6yz$$

$\therefore$ बिन्दु (1, −1, 1) पर div $\vec{\mathbf{F}} = (-1)^2 + 2\times(1)^2\times(1) - 6\times(-1)\times1 = 1+2+6 = 9$

$\therefore$ $\vec{\mathbf{F}} = \nabla(x^3+y^3+z^3-3xyz)$

$$= \hat{\mathbf{i}}\frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz) + \hat{\mathbf{j}}\frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz) + \hat{\mathbf{k}}\frac{\partial}{\partial z}(x^3+y^3+z^3-3xyz)$$

$$= \hat{\mathbf{i}}(3x^2-3yz) + \hat{\mathbf{j}}(3y^2-3xz) + \hat{\mathbf{k}}(3z^2-3xy)$$

$$= 3[(x^2-yz)\,\hat{\mathbf{i}} + (y^2-xz)\,\hat{\mathbf{j}} + (z^2-xy)\,\hat{\mathbf{k}}]$$

$\therefore$ div $\vec{\mathbf{F}} = \nabla\,.\,\vec{\mathbf{F}} = 3\left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)[(x^2-yz)\,\hat{\mathbf{i}} + (y^2-xz)\,\hat{\mathbf{j}} + (z^2-xy)\,\hat{\mathbf{k}}]$

$$= 3\left[\frac{\partial}{\partial x}(x^2-yz) + \frac{\partial}{\partial y}(y^2-xz) + \frac{\partial}{\partial z}(z^2-xy)\right]$$

$$= 3[2x+2y+2z] = 6(x+y+z)$$

(i) $\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \Rightarrow r^2 = x^2+y^2+z^2$

अतः $2r\frac{\partial r}{\partial x} = 2x$ or $\frac{\partial r}{\partial x} = \frac{x}{r}$

इसी तरह $\frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial z}{\partial r} = \frac{z}{r}$ ...(i)

$$\text{grad } r^n = \nabla r^n = \hat{\mathbf{i}}\frac{\partial r^n}{\partial x} + \hat{\mathbf{j}}\frac{\partial r^n}{\partial y} + \hat{\mathbf{k}}\frac{\partial r^n}{\partial z} = \hat{\mathbf{i}}\,nr^{n-1}\frac{\partial r}{\partial x} + \hat{\mathbf{j}}\,nr^{n-1}\frac{\partial r}{\partial y} + \hat{\mathbf{k}}n\frac{\partial r^{n-1}}{\partial z}$$

$$= nr^{n-1}\left[\frac{x}{r}\hat{\mathbf{i}} + \frac{y}{r}\hat{\mathbf{j}} + \frac{z}{r}\hat{\mathbf{k}}\right] = n\frac{r^{n-1}}{r}[x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}] = nr^{n-2}\,\vec{\mathbf{r}}$$

(ii) दिया गया $r = |\vec{\mathbf{r}}|$ तथा $\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

$\therefore$ $r = (x^2+y^2+z^2)^{1/2}$

$\therefore$ $\nabla r = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{1/2}$

$$= \hat{\mathbf{i}}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2} + \hat{\mathbf{j}}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{1/2} + \hat{\mathbf{k}}\frac{\partial}{\partial z}(x^2+y^2z^2)^{1/2}$$

$$= \hat{\mathbf{i}}\times\frac{1}{2}\frac{2x}{(x^2+y^2+z^2)^{1/2}} + \hat{\mathbf{j}}\frac{1}{2}\times\frac{2y}{(x^2+y^2+z^2)^{1/2}} + \hat{\mathbf{k}}\frac{1}{2}\times\frac{2z}{(x^2+y^2+z^2)^{1/2}}$$

$$= \frac{1}{(x^2+y^2+z^2)^{1/2}}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) = \frac{1}{r}\vec{\mathbf{r}}$$

5. माना $\phi = 4xz^3 - 3x^2y^2z^2$

तथा z-अक्ष की दिशा में इकाई सदिश $= 0\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 1\hat{\mathbf{k}} = \hat{\mathbf{k}}$

अब $\text{grad } \phi = \nabla\phi = \hat{\mathbf{i}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{j}}\frac{\partial\phi}{\partial y} + \hat{\mathbf{k}}\frac{\partial\phi}{\partial z}$

$$= \hat{\mathbf{i}}\frac{\partial}{\partial x}(4xz^3 - 3x^2y^2z^2) + \hat{\mathbf{j}}\frac{\partial}{\partial y}(4xz^3 - 3x^2y^2z^2) + \hat{\mathbf{k}}\frac{\partial}{\partial z}(4xz^3 - 3x^2y^2z^2)$$

$$= \hat{\mathbf{i}}(4z^3 - 6xy^2z^2) + \hat{\mathbf{j}}(-6x^2yz^2) + \hat{\mathbf{k}}(12xz^2 - 6x^2y^2z) \quad ...($$

(1) से $x = 2, y = -1$ तथा $z = 2$ रखने पर

$$\text{grad } \phi = \hat{\mathbf{i}}[4\times(2)^3 - 6\times 2(-1)^2\times(2)^2] + \hat{\mathbf{j}}[-6\times(2)^2(-1)\times(2)^2]$$
$$+ \hat{\mathbf{k}}[12\times(2)(2)^2 - 6\times(2)\times(-1)^2(2)] = -16\hat{\mathbf{i}} + 96\hat{\mathbf{j}} + 7$$

$\therefore$ अभीष्ट दिष्ठ अवकलज (directional derivative) बिन्दु $(2, -1, 2)$ पर z-अक्ष दिशा में $\nabla\phi.\hat{\mathbf{r}} = \nabla\phi.\hat{\mathbf{k}}$

$$= (-16\hat{\mathbf{i}} + 96\hat{\mathbf{j}} + 72\hat{\mathbf{k}}).\hat{\mathbf{k}} = 72 \qquad [\because \hat{\mathbf{i}}.\hat{\mathbf{j}} = \hat{\mathbf{j}}.\hat{\mathbf{k}} = \mathbf{0}]$$

6. दी गई सरल रेखा $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$

अत: 2, 2, 1 इस रेखा के दिक् कोज्या (d.c) हैं। माना $\phi = x^2 + y^2 + z^2$

तथा $\vec{\mathbf{a}} = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ सदिश ϕ की दिशा में तब

$$\hat{\mathbf{a}} = \frac{\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|} = \frac{2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{4+4+1}}$$

$$\hat{\mathbf{a}} = \frac{2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}}{3} \quad ...(1$$

अब $\text{grad } \phi = \nabla\phi = \hat{\mathbf{i}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{j}}\frac{\partial\phi}{\partial y} + \hat{\mathbf{k}}\frac{\partial\phi}{\partial z}$

$$= \hat{\mathbf{i}}(2x) + \hat{\mathbf{j}}(2y) + \hat{\mathbf{k}}(2z) = 2(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \quad ...(2$$

अत: दिये गये दिशा में दिष्टगुणज

$$= \vec{\mathbf{a}} \text{ grad } \phi = \frac{\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|}.\nabla\phi = \left(\frac{2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}}{3}\right).2(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \quad \text{[समी॰ (1) तथा (2) से]}$$

$$= \frac{4x + 4y + 2z}{3}$$

$\therefore$ बिन्दु $x = 1, y = 2, z = 3$

बिन्दु पर अभीष्ट दिष्टगुणज $= \frac{4\times 1 + 4\times 2 + 2\times 3}{3} = \frac{4+8+6}{3} = \frac{18}{3} = 6$

7. (i) प्रश्न से $\vec{\mathbf{F}} = (x + y + 1)\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}(-x - y) \quad ...(1$

$$\text{curl } \vec{\mathbf{F}} = \nabla\times\vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + y + 1 & 1 & -x - y \end{vmatrix}$$

$$= \left\{\left(\frac{\partial}{\partial y}(-x - y) - \frac{\partial}{\partial z}(1)\right)\right\}\hat{\mathbf{i}} - \hat{\mathbf{j}}\left\{\left(\frac{\partial}{\partial x}(-x - y)\right) - \frac{\partial}{\partial z}(x + y + 1)\right\}$$

$$+ \hat{\mathbf{k}}\left\{\frac{\partial}{\partial x}(1) - \frac{\partial}{\partial y}(x + y + 1)\right.$$

$$= -\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}} \qquad \ldots(2)$$

$\therefore$ $\vec{\mathbf{F}} \cdot \text{curl}\ \vec{\mathbf{F}} = \vec{\mathbf{F}} \cdot (-\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}) = [\hat{\mathbf{i}}(x+y+1) + \hat{\mathbf{j}} + \hat{\mathbf{k}}(-x-y)] \cdot [-\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}]$

[(1) तथा (2) से]

$$= -(x+y+1) + 1 + x + y = 0$$

$[\because \hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1, \hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0]$

$\therefore$ $\vec{\mathbf{F}} \cdot \text{curl}\ \vec{\mathbf{F}} = 0$

(ii) प्रश्न से $\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

$\therefore$ $|\vec{\mathbf{r}}| = r = \sqrt{x^2 + y^2 + z^2}$ तथा $\phi = \frac{1}{r} = r^{-1} = (x^2 + y^2 + z^2)^{-1/2}$

$\therefore$ $\nabla\phi = \nabla\,(x^2 + y^2 + z^2)^{-1/2}$

$$= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)^{-1/2}$$

$$= \hat{\mathbf{i}}\frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{-1/2} + \hat{\mathbf{j}}\frac{\partial}{\partial y}(x^2 + y^2 + z^2)^{-1/2} + \hat{\mathbf{k}}\frac{\partial}{\partial z}(x^2 + y^2 + z^2)^{-1/2}$$

$$= -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2x)\,\hat{\mathbf{i}} + \hat{\mathbf{j}}\left(-\frac{1}{2}\right)(x^2 + y^2 + z^2)^{-3/2}(2y) + \hat{\mathbf{k}}\left(-\frac{1}{2}\right)(x^2 + y^2 + z^2)^{-3/2}(2z)$$

$$= -\frac{1}{2} \times 2\,\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{(x^2 + y^2 + z)^{3/2}} = -\frac{\vec{\mathbf{r}}}{r^3},$$ जहाँ $\vec{\mathbf{r}} = x\hat{\mathbf{i}} + + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

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