अध्याय

7

सदिश अवकल[illegible] एवं समाकल[illegible]

(Vector Differentiation and Integratio

7.1 सदिश फलनों के अवकलन के महत्वपूर्ण नियम (Important Rules of Differentiation of Vector Functions)

1. यदि $\vec{\mathbf{F}}(t) = F_1(t)\,\vec{\mathbf{i}} + F_2(t)\,\vec{\mathbf{j}} + F_3(t)\,\vec{\mathbf{k}}$ तो

$$\frac{d}{dt}\vec{\mathbf{F}}(t) = \frac{d}{dt}F_1(t)\,\vec{\mathbf{i}} + \frac{d}{dt}F_2(t)\,\vec{\mathbf{j}} + \frac{d}{dt}F_3(t)\,\vec{\mathbf{k}}$$

2. $\frac{d}{dt}(\vec{\mathbf{F}} \pm \vec{\mathbf{G}}) = \frac{d\vec{\mathbf{F}}}{dt} \pm \frac{d\vec{\mathbf{G}}}{dt}$

3. $\frac{d}{dt}(\phi\,\vec{\mathbf{F}}) = \phi\frac{d}{dt}\vec{\mathbf{F}}$

4. $\frac{d}{dt}(\vec{\mathbf{F}} \cdot \vec{\mathbf{G}}) = \vec{\mathbf{F}} \cdot \frac{d\vec{\mathbf{G}}}{dt} + \frac{d\vec{\mathbf{F}}}{dt} \cdot \vec{\mathbf{G}}$

5. $\frac{d}{dt}(\vec{\mathbf{F}} \times \vec{\mathbf{G}}) = \vec{\mathbf{F}} \times \frac{d\vec{\mathbf{G}}}{dt} + \frac{d\vec{\mathbf{F}}}{dt} \times \vec{\mathbf{G}}$ [$\vec{\mathbf{F}}$ तथा $\vec{\mathbf{G}}$ का क्रम परिवर्तित नहीं होगा]

6. (i) यदि $\vec{\mathbf{r}}$, किसी सदिश चर t का अवकलनीय सदिश फलन है, जहाँ t दूसरे चर s का फलन है, तो

$$\frac{d\vec{\mathbf{r}}}{dt} = \frac{d\vec{\mathbf{r}}}{ds} \times \frac{ds}{dt}$$

(ii) वेग $\vec{\mathbf{V}} = \frac{d\vec{\mathbf{r}}}{dt}$; त्वरण $\vec{\mathbf{a}} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{d^2\vec{\mathbf{r}}}{dt^2}$, जहाँ $\vec{\mathbf{r}}$ समय t पर बिन्दु स्थिति सदिश है।

7.2 सदिश फलनों का अनिश्चित समाकलन (Indefinite Integration of Vector Functions)

माना $\vec{\mathbf{f}}(t)$ तथा $\vec{\mathbf{F}}(t)$ अदिश चर t के दो सदिश फलन हैं तथा $\frac{d}{dt}\vec{\mathbf{F}}(t) = \vec{\mathbf{f}}(t)$ तो $\vec{\mathbf{F}}(t)$ को $\vec{\mathbf{f}}(t)$ अनिश्चित समाकल कहा जाता है तथा इसे निम्न रूप में व्यक्त करते हैं :

$$\int \vec{\mathbf{f}}(t)\,dt = \vec{\mathbf{F}}(t) + \vec{\mathbf{c}}$$

7.2.1 निश्चित समाकलन

यदि समाकलन की निम्न सीमा $t = a$ तथा उच्च सीमा $t = b$ हो तो

$$\int_a^b \vec{\mathbf{f}}(t)\,dt = \left[\vec{\mathbf{F}}(t)\right]_a^b = \vec{\mathbf{F}}(b) - \vec{\mathbf{F}}(a)$$ को निश्चित समाकलन कहते है।

(i) $\int \frac{d}{dt}(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})\,dt = \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + c$

(ii) $\int\left(\vec{u}\times\frac{d\vec{v}}{dt}+\vec{v}=\frac{d\vec{u}}{dt}\times\vec{v}\right)dt=\int\frac{d}{dt}(\vec{u}\times\vec{v})=\vec{u}\times\vec{v}+\vec{c}$

(iii) $\int\left\{2\vec{u}\cdot\frac{d\vec{u}}{dt}\right\}dt=u^2+c$ (iv) $\int\left(2\frac{d\vec{u}}{dt}\cdot\frac{d^2\vec{u}}{dt}\right)dt=\left(\frac{d\vec{u}}{dt}\right)^2+0$

(v) $\int\left(\vec{a}\times\frac{d\vec{u}}{dt}\right)dt=\vec{a}\times\vec{u}+\vec{c}$ (vi) $\int\left(\vec{a}\cdot\frac{d\vec{u}}{dt}\right)dt=\vec{a}\cdot\vec{u}+c$

(vii) $\int\left\{\vec{u}\times\frac{d^2\vec{u}}{dt}\right\}dt=\vec{u}\times\frac{d\vec{u}}{dt}+\vec{c}$ (viii) $\int\left[\frac{1}{|\vec{u}|}\frac{d\vec{u}}{dt}\frac{1}{|\vec{u}|}\vec{u}\right]dt=\frac{\vec{u}}{|\vec{u}|}+\vec{c}=\hat{u}+\vec{c}$

(ix) $\vec{f}(t)=f_1(t)\vec{i}+f_2(t)\vec{j}+f_3(t)\vec{k}$ तो

$\int\vec{f}(t)\,dt=\vec{i}\int f_1(t)\,dt+\vec{j}\int f_2(t)\,dt+\vec{k}\int f_3(t)\,dt$, जहाँ $\vec{c}$ सदिश अचर; c अदिश अचर

परीक्षा में पूछे गए एवं अन्य महत्वपूर्ण प्रश्न

✍ वस्तुनिष्ठ प्रश्न

1. यदि $\mathbf{r}=\sin t\mathbf{i}+\cos t\mathbf{j}$ तो $\frac{d\mathbf{r}}{dt}=$

(a) $\cos t-\sin t$ (b) $\cos t\mathbf{i}+\sin t\mathbf{j}$ (c) $\cos t\mathbf{i}-\sin t\mathbf{j}$ (d) कोई नहीं

2. यदि $\mathbf{r}$ कोई सदिश हो तथा $|\mathbf{r}|=r$ तो $\frac{d}{dt}(\mathbf{r}.\mathbf{r})=$

(a) $2\mathbf{r}$ (b) $2r\frac{d\mathbf{r}}{dt}$ (c) $\mathbf{r}.\frac{d\mathbf{r}}{dt}$ (d) कोई नहीं

3. यदि $\mathbf{r}$ कोइ ʽ सदिशहो तो $\frac{d}{dt}(\mathbf{r}\times\mathbf{r})=$

(a) $2\mathbf{r}\frac{d\mathbf{r}}{dt}$ (b) $\mathbf{r}\times\frac{d\mathbf{r}}{dt}$ (c) 0 (d) कोई नहीं

4. यदि $f(t)=(t-t^2)\mathbf{i}+2t^3\mathbf{j}-3\mathbf{k}$ तो $\int\mathbf{f}(t)\,dt=$

(a) $\frac{t^3}{2^3}\mathbf{i}+\frac{t^4}{4^2}\mathbf{j}-3t\mathbf{k}+c$ (b) $\left(\frac{t^2}{2}-\frac{t^3}{3}\right)\mathbf{i}+\frac{t^4}{2}\mathbf{j}-3t\mathbf{k}+c$

(c) $\frac{t^2}{2}\mathbf{i}+\frac{t^4}{2}\mathbf{j}-3\mathbf{k}+c$ (d) कोई नहीं

5. $\int_0^1(e^t\mathbf{i}+e^{-2t}\mathbf{j}+t\mathbf{k})\,dt=$

(a) $e\mathbf{i}+e^{-2}\mathbf{j}+\mathbf{k}$ (b) $(e-1)\,\mathbf{i}-\frac{1}{2}(e^{-2}-1)\mathbf{j}+\frac{1}{2}\mathbf{k}$

(c) $(1-e)\mathbf{i}+\frac{1}{2}(e^{-2}+1)\mathbf{j}+\frac{1}{2}\mathbf{k}$ (d) कोई नहीं

✍ अति लघुउत्तरीय प्रश्न

1. यदि $\vec{r}=\sin t\,\vec{i}+\cos t\,\vec{j}+t\,\vec{k}$, तो निम्नलिखित के मान निकालें—

(i) $\frac{d\vec{\mathbf{r}}}{dt}$ (ii) $\frac{d^2\vec{\mathbf{r}}}{dt^2}$

2. यदि $\vec{\mathbf{r}}$ अदिश t का सदिश फलन हो तथा $\vec{\mathbf{A}}$ एक अचर सदिश हो तो निम्न के t सापेक्ष अवकलन निकालें—

(i) $\vec{\mathbf{r}}\cdot\vec{\mathbf{A}}$ (ii) $\vec{\mathbf{r}}\times\vec{\mathbf{A}}$ (iii) $\vec{\mathbf{r}}\cdot\frac{d\vec{\mathbf{r}}}{dt}$ (iv) $\vec{\mathbf{r}}\times\frac{d\vec{\mathbf{r}}}{dt}$

3. यदि $\vec{\mathbf{f}}(t)=(3t^3-t)\vec{\mathbf{i}}+(2t^2-5)\vec{\mathbf{j}}-4t\,\vec{\mathbf{k}}$ तो $\int\vec{\mathbf{f}}(t)\,dt$ का मान निकालें।

4. यदि $\vec{\mathbf{f}}(t)=5t^4\,\vec{\mathbf{i}}+8t^3\,\vec{\mathbf{j}}-4t^2\,\vec{\mathbf{k}}$ तो $\int\vec{\mathbf{f}}(t)\,dt$ का मान बताये।।

लघुउत्तरीय प्रश्न/दीर्घउत्तरीय प्रश्न

1. (a) यदि $\vec{\mathbf{a}}(t)=t\hat{\mathbf{i}}-t^2\hat{\mathbf{j}}+(t-1)\hat{\mathbf{k}}$ और $\vec{\mathbf{b}}(t)=2t^2\hat{\mathbf{i}}+6t\hat{\mathbf{k}}$ सिद्ध कीजिए कि

$$\int_0^1(\vec{\mathbf{a}}\times\vec{\mathbf{b}})\,dt=-\frac{3}{2}\hat{\mathbf{i}}-\frac{13}{6}\hat{\mathbf{j}}+\frac{2}{5}\hat{\mathbf{k}}$$

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(b) यदि $\vec{\mathbf{r}}$ scalar t का Vector Function है तथा r इसके Modulus एवं a, b सदिश के स्थिरांक हैं $r^2.\vec{\mathbf{r}}+(a.r)\,b$ तब निम्न को t के सापेक्ष अवकलित करें।

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(c) यदि $r=|\vec{\mathbf{r}}|$ जहाँ $r=x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}$, तो सिद्ध करो $\nabla r=\frac{1}{r}\vec{\mathbf{r}}$.

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(d) यदि $r=t^3\hat{\mathbf{i}}+\left(2t^3-\frac{1}{5t^2}\right)\hat{\mathbf{j}}$, तो दर्शाइये कि $r\times\frac{dr}{dt}=\hat{\mathbf{k}}$

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(e) यदि $\vec{\mathbf{r}}=a\cos t\hat{\mathbf{i}}+a\sin t\hat{\mathbf{j}}+at\tan a\hat{\mathbf{k}}$ तो दर्शाइये कि $\frac{d\vec{\mathbf{r}}}{dt}\times\frac{d^2\vec{\mathbf{r}}}{dt^2}$

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(f) यदि $\vec{\mathbf{a}},\vec{\mathbf{b}}$ अचर सदिश हैं, ω अचर है तथा $\vec{\mathbf{r}}$, t का सदिश फलन है, जो $\vec{\mathbf{r}}=\cos\omega t\,\vec{\mathbf{a}}+\sin\omega t\,\vec{\mathbf{b}}$ से दि जाता है, तो दिखायें कि

(i) $\frac{d^2\vec{\mathbf{r}}}{dt^2}+\omega^2\vec{\mathbf{r}}=0$ (ii) $\vec{\mathbf{r}}\times\frac{d\vec{\mathbf{r}}}{dt}=\omega(\vec{\mathbf{a}}\times\vec{\mathbf{b}})$

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2. यदि $\vec{\mathbf{r}}=\log(1+t^2)\,\vec{\mathbf{i}}+\sin t\,\vec{\mathbf{j}}-t^2\,\vec{\mathbf{k}}$, तो $\frac{d\vec{\mathbf{r}}}{dt}$ का मान $t=0$ पर ज्ञात करें।

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3. यदि $\vec{\mathbf{r}}=\vec{\mathbf{a}}\,e^{nt}+\vec{\mathbf{b}}\,e^{-nt}$, जहाँ $\vec{\mathbf{a}}$ तथा $\vec{\mathbf{b}}$ अचर सदिश है, तो सिद्ध करें $\frac{d^2\vec{\mathbf{r}}}{dt^2}-n^2\vec{\mathbf{r}}=0$

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4. यदि $\frac{d\vec{\mathbf{u}}}{dt}=\vec{\mathbf{w}}\times\vec{\mathbf{u}}$, $\frac{d\vec{\mathbf{v}}}{dt}=\vec{\mathbf{w}}\times\vec{\mathbf{v}}$, तो दिखायें कि $\frac{d}{dt}(\vec{\mathbf{u}}\times\vec{\mathbf{v}})=\vec{\mathbf{w}}\times(\vec{\mathbf{u}}\times\vec{\mathbf{v}})$

5. यदि $\vec{\mathbf{a}}=\sin\theta\,\vec{\mathbf{i}}+\cos\theta\,\vec{\mathbf{j}}+\theta\,\vec{\mathbf{k}}$; $\vec{\mathbf{b}}=\cos\theta\,\vec{\mathbf{i}}-\sin\theta\,\vec{\mathbf{j}}-3\,\vec{\mathbf{k}}$; $\vec{\mathbf{c}}=2\,\vec{\mathbf{i}}+3\,\vec{\mathbf{j}}-\vec{\mathbf{k}}$, तब $\theta=0$ पर $\frac{d}{d\theta}\{[\vec{\mathbf{a}}\times(\vec{\mathbf{b}}\times\vec{\mathbf{c}})]\}$ का मान ज्ञात कीजिये।

6. यदि $\vec{\mathbf{r}}=\cos nt\,\hat{\mathbf{i}}+\sin nt\,\hat{\mathbf{j}}$ जहाँ n अचर तो दिखायें

(i) $\vec{\mathbf{r}}\times\frac{d\vec{\mathbf{r}}}{dt}=n\hat{\mathbf{k}}$ (ii) $\frac{d^2\vec{\mathbf{r}}}{dt}=-n^2\vec{\mathbf{r}}$

7. यदि $\vec{r}(t) = 5t^2\,\vec{i} + t\,\vec{j} - t^3\,\vec{k}$ तो सिद्ध करें कि $\int_1^2 \vec{r} \times \frac{d^2\vec{r}}{dt^2}\,dt = -14\,\vec{i} + 75\,\vec{j} - 15\,\vec{k}$

8. $\int_0^1 (e^{3t}\,\vec{i} + e^{-2t}\,\vec{j} + e^{4t}\,\vec{k})\,dt$ का मान निकालें।

9. यदि $\vec{r}.(t) = t\,\vec{i} - t^2\,\vec{j} + (t-1)\,\vec{k}$ तथा $\vec{s}(t) = 2t^2\,\vec{i} + 6t\,\vec{k}$

तो (i) $\int_0^1 \vec{r}\,.\,\vec{s}\,dt$ तथा (ii) $\int_0^2 \vec{r} \times \vec{s}\,dt$ का मान बतायें।

10. समाकलित करें $\vec{a} \times \frac{d^2\vec{r}}{dt^2} = \vec{b}$

11. $\int_0^3 \int_0^2 \int_0^1 (x+y+z)\,dz\,dx\,dy$ का मान ज्ञात करें। [2007]

12. $\int_1^2 \int_0^{y/2} y\,dy\,dx$ का मान ज्ञात करें। [2005]

13. $\int_0^b \int_0^a (x^2+y^2)\,dx\,dy$ का मान ज्ञात करें। [2004]

14. $\int_0^1 \int_0^1 \int_0^1 e^{x+y+z}\,dx\,dy\,dz$ का मान ज्ञात करें। [2018(S)]

हल एवं संकेत

वस्तुनिष्ठ प्रश्न

1. (c) 2. (b) 3. (c) 4. (b) 5. (b)

अति लघुउत्तरीय प्रश्न

1. अब (i) $\frac{d\vec{r}}{dt} = \frac{d}{dt}(\sin t)\,\vec{i} + \frac{d}{dt}(\cos t)\,\vec{j} + \frac{d}{dt}(t)\,\vec{k} = \cos t\,\vec{i} - \sin t\,\vec{j} + \vec{k}$ उत्तर

$\left[\because \frac{d\vec{i}}{dt} = \frac{d\vec{j}}{dt} = \frac{d\vec{k}}{dt} = 0\right]$

(ii) $\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d}{dt}(\cos t)\,\vec{i} - \frac{d}{dt}(\sin t)\,\vec{j} + \frac{d\vec{k}}{dt}$

$= -\sin t\,\vec{i} - \cos t\,\vec{j} + 0 = -\sin t\,\vec{i} - \cos\,\vec{j}$ उत्तर

2. (i) $\frac{d}{dt}(\vec{r}\,.\,\vec{A}) = \frac{d\vec{r}}{dt}\,.\,\vec{A} + \vec{r}\,.\,\frac{d\vec{A}}{dt} = \frac{d\vec{r}}{dt}\,.\,\vec{A}$ $\left[\because \frac{d\vec{A}}{dt} = 0\right]$

(ii) $\frac{d}{dt}(\vec{r} \times \vec{A}) = \frac{d\vec{r}}{dt} \times \vec{A} + \vec{r} \times \frac{d\vec{A}}{dt} = \frac{d\vec{r}}{dt} \times \vec{A}$ $\left[\because \frac{d\vec{A}}{dt} = 0\right]$

(iii) $\frac{d}{dt}\left(\vec{r}\,.\,\frac{d\vec{r}}{dt}\right) = \frac{d\vec{r}}{dt}\,.\,\frac{d\vec{r}}{dt} + \vec{r}\,.\,\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \left(\frac{d\vec{r}}{dt}\right)^2 + \vec{r}\,.\left(\frac{d^2\vec{r}}{dt^2}\right)$

(iv) $\frac{d}{dt}\left(\vec{\mathbf{r}} \times \frac{d\vec{\mathbf{r}}}{dt}\right) = \frac{d\vec{\mathbf{r}}}{dt} \times \frac{d\vec{\mathbf{r}}}{dt} + \vec{\mathbf{r}} \times \frac{d}{dt}\left(\frac{d\vec{\mathbf{r}}}{dt}\right) = \vec{\mathbf{r}} \times \frac{d^2\vec{\mathbf{r}}}{dt^2} = \mathbf{r} \times \frac{d^2\vec{\mathbf{r}}}{dt^2}$ $[\because \dot{\mathbf{r}} \times \dot{\mathbf{r}} = 0]$

3. $\int \vec{\mathbf{f}}(t)\,dt = \int \left\{(3t^3 - t)\,\vec{\mathbf{i}} + (2t^2 - 5)\,\vec{\mathbf{j}} - 4t\,\vec{\mathbf{k}}\right\} dt$

$$= \left(\frac{3}{4}t^4 - \frac{t^2}{2}\right)\vec{\mathbf{i}} + \left(\frac{2}{3}t^3 - 5t\right)\vec{\mathbf{j}} - \frac{4t^2}{2}\vec{\mathbf{k}} + \vec{\mathbf{c}}$$

$$= \left(\frac{3}{4}t^4 - \frac{t^2}{2}\right)\vec{\mathbf{i}} + \left(\frac{2}{3}t^3 - 5t\right)\vec{\mathbf{j}} - 2t^2\,\vec{\mathbf{k}} + \vec{\mathbf{c}}$$

4. $\int \vec{\mathbf{f}}(t)\,dt = \int (5t^4\,\vec{\mathbf{i}} + 8t^3\,\vec{\mathbf{j}} - 4\vec{t}\,\vec{\mathbf{k}})\,dt = 5 \times \frac{t^5}{5}\vec{\mathbf{i}} + 8\frac{t^4}{4}\vec{\mathbf{j}} - 4\frac{t^3}{3}\vec{\mathbf{k}} + \vec{\mathbf{c}}$

$$= t^5\,\vec{\mathbf{i}} + 2t^4\,\vec{\mathbf{j}} - \frac{4}{3}t^3\,\vec{\mathbf{k}} + \vec{\mathbf{c}}$$

✍ लघुउत्तरीय प्रश्न/दीर्घउत्तरीय प्रश्न

1. (a) यहाँ $\vec{\mathbf{a}}(t) = t\hat{\mathbf{i}} - t^2\hat{\mathbf{j}} + (t-1)\hat{\mathbf{k}},\ \vec{\mathbf{b}}(t) = 2t^2\hat{\mathbf{i}} + 6t\hat{\mathbf{k}}$

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ t & -t^2 & t-1 \\ 2t^2 & 0 & 6t \end{vmatrix} = \hat{\mathbf{i}}\begin{vmatrix} -t^2 & t-1 \\ 0 & 6t \end{vmatrix} - \hat{\mathbf{j}}\begin{vmatrix} t & t-1 \\ 2t^2 & 6t \end{vmatrix} + \hat{\mathbf{k}}\begin{vmatrix} t & -t^2 \\ 2t^2 & 0 \end{vmatrix}$$

$$= \hat{\mathbf{i}}\,(-6t^3 - 0) - \hat{\mathbf{j}}\,\{6t^2 - 2t^2(t-1)\} + \hat{\mathbf{k}}\,(0 - 2t^2)$$

$$= -6t^3\hat{\mathbf{i}} - \hat{\mathbf{j}}\,(6t^2 - 2t^3 + 2t^2) + 2t^4\hat{\mathbf{k}}$$

$$= -6t^3\hat{\mathbf{i}} + (2t^3 - 8t^2)\,\hat{\mathbf{j}} + 2t^4\,\hat{\mathbf{k}}$$

$\therefore \quad \int_0^1 \vec{\mathbf{a}} \times \vec{\mathbf{b}}\,dt = \int_0^1 \{-6t^3\hat{\mathbf{i}} + (2t^3 - 8t^2)\,\hat{\mathbf{j}} + 2t^4\}\,dt$

$$= -6\hat{\mathbf{i}} \times \left(\frac{t^4}{4}\right)_0^1 + 2\hat{\mathbf{j}}\left(\frac{t^4}{4} - 4 \times \frac{t^3}{3}\right)_0^1 + 2\left[\frac{t^5}{5}\right]_0^1$$

$$= -\frac{6}{4}(1^4 - 0)\,\hat{\mathbf{i}} + 2\left(\frac{1}{4} - 4 \times \frac{1^3}{3}\right)\hat{\mathbf{j}} + 2\left(\frac{1^5}{5} - 0\right)\hat{\mathbf{k}}$$

$$= -\frac{3}{2}\hat{\mathbf{i}} - \left(\frac{1}{2} - \frac{8}{3}\right)\hat{\mathbf{j}} + \frac{2}{5}\hat{\mathbf{k}}$$

$$= -\frac{3}{2}\hat{\mathbf{i}} + \left(\frac{3-16}{6}\right)\hat{\mathbf{j}} + \frac{2}{5}\hat{\mathbf{k}}$$

$$= -\frac{3}{2}\hat{\mathbf{i}} - \frac{13}{6}\hat{\mathbf{j}} + \frac{2}{5}\hat{\mathbf{k}}$$

(b) माना $\vec{A} = r^2\,\vec{\mathbf{r}} + (\vec{\mathbf{a}}\,.\,\vec{\mathbf{r}})\,\vec{\mathbf{b}}$

तो $\frac{d\vec{A}}{dt} = \frac{d}{dt}(r^2\,\vec{\mathbf{r}}) + \frac{d}{dt}\{(\vec{\mathbf{a}}\,.\,\vec{\mathbf{r}})\,\vec{\mathbf{b}}\}$

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$$= 2r\frac{dr}{dt}\vec{\mathbf{r}} + r^2\frac{d\vec{\mathbf{r}}}{dt} + \left\{\frac{d}{dt}(\vec{\mathbf{a}}\cdot\vec{\mathbf{r}})\vec{\mathbf{b}}\right\} + (\vec{\mathbf{a}}\cdot\vec{\mathbf{r}})\frac{d\vec{\mathbf{b}}}{dt}$$

$$= 2r\frac{dr}{dt}\vec{\mathbf{r}} + r^2\frac{d\vec{\mathbf{r}}}{dt} + \left\{\frac{d\vec{\mathbf{a}}}{dr}\cdot\vec{\mathbf{r}} + \vec{\mathbf{a}}\cdot\frac{d\vec{\mathbf{r}}}{dr}\right\}\vec{\mathbf{b}} \qquad \left[\because \frac{d\vec{\mathbf{b}}}{dt} = 0\right]$$

$$= 2r\frac{dr}{dt}\vec{\mathbf{r}} + r^2\frac{d\vec{\mathbf{r}}}{dt} + \left(\vec{\mathbf{a}}\cdot\frac{d\vec{\mathbf{r}}}{dt}\right)\vec{\mathbf{b}} \qquad \left[\because \frac{d\vec{\mathbf{a}}}{dt} = 0\right]$$

(c) दिया गया $r = |\vec{\mathbf{r}}|$ तथा $\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ $\quad\therefore\quad r = (x^2 + y^2 + z^2)^{1/2}$

$$\therefore\quad \nabla r = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)^{1/2}$$

$$= \hat{\mathbf{i}}\frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{1/2} + \hat{\mathbf{j}}\frac{\partial}{\partial y}(x^2 + y^2 + z^2)^{1/2} + \hat{\mathbf{k}}\frac{\partial}{\partial z}(x^2 + y^2 z^2)^{1/2}$$

$$= \hat{\mathbf{i}}\times\frac{1}{2}\frac{2x}{(x^2 + y^2 + z^2)^{1/2}} + \hat{\mathbf{j}}\frac{1}{2}\times\frac{2y}{(x^2 + y^2 + z^2)^{1/2}} + \hat{\mathbf{k}}\frac{1}{2}\times\frac{2z}{(x^2 + y^2 + z^2)^{1/2}}$$

$$= \frac{1}{(x^2 + y^2 + z^2)^{1/2}}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) = \frac{1}{r}\vec{\mathbf{r}}$$

(d) $\vec{\mathbf{r}} = t^3\hat{\mathbf{i}} + \left(2t^3 - \frac{1}{5t^2}\right)\hat{\mathbf{j}}$

$$\therefore\quad \frac{d\vec{\mathbf{r}}}{dt} = 3t^2\hat{\mathbf{i}} + \left(6t^2 - \frac{1}{5}\times(-2)\,t^{-2-1}\right)\hat{\mathbf{j}} = 3t^2\hat{\mathbf{i}} + \left(6t^2 + \frac{2}{5t^3}\right)\hat{\mathbf{j}}$$

$$\therefore\quad \vec{\mathbf{r}}\times\frac{d\vec{\mathbf{r}}}{dt} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ t^3 & 2t^3 - \dfrac{1}{5t^2} & 0 \\ 3t^2 & 6t^2 + \dfrac{2}{5t^2} & 0 \end{vmatrix}$$

$$= \hat{\mathbf{i}}(0-0) - \hat{\mathbf{j}}(0-0) + \hat{\mathbf{k}}\left\{t^3\left(6t^2 + \frac{2}{5t^3}\right) - 3t^2\left(2t^3 - \frac{1}{5t^3}\right)\right\}$$

$$= 0 - 0 + \hat{\mathbf{k}}\left\{6t^5 + \frac{2}{5} - 6t^5 + \frac{3}{5}\right\} = \hat{\mathbf{k}}\times\left\{\frac{2}{5} + \frac{3}{5}\right\} = \hat{\mathbf{k}}\times 1 = \hat{\mathbf{k}}$$

(e) दिया गया $\vec{\mathbf{r}} = a\cos t\hat{\mathbf{i}} + a\sin t\hat{\mathbf{j}} + at\tan a\hat{\mathbf{k}}$

$$\frac{d\vec{\mathbf{r}}}{dt} = -a\sin t\hat{\mathbf{i}} + a\cos t\hat{\mathbf{j}} + a\tan a\hat{\mathbf{k}} \qquad ...(1)$$

समी० (1) का t के सापेक्ष अवकलन से

$$\frac{d\vec{\mathbf{r}}}{dt} = -a\sin t\hat{\mathbf{i}} - a\cos t\hat{\mathbf{j}} + 0.\hat{\mathbf{k}}$$

$$\frac{d\vec{\mathbf{r}}}{dt}\times\frac{d^2\vec{\mathbf{r}}}{dt^2}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ -a\sin t & a\cos t & a\tan\alpha\\ -a\cos t & -a\sin t & 0\end{vmatrix}$$

$$=\hat{\mathbf{i}}(0+a^2\sin\tan\alpha)-\hat{\mathbf{j}}\,(0+a^2\cos t\tan\alpha)+\hat{\mathbf{k}}(a^2\sin^2 t+a^2\cos^2 t$$

$$=a^2\sin t\tan\alpha\hat{\mathbf{i}}-a^2\cos t\tan\alpha\hat{\mathbf{j}}+a^2\,\hat{\mathbf{k}}(\sin^2 t+\cot^2 t)$$

$$=a^2\tan\alpha\,(\hat{\mathbf{i}}\sin t-\hat{\mathbf{j}}\cos t)+a^2\hat{\mathbf{k}}$$

(f) (i) $\because \vec{\mathbf{r}}=\cos\omega t\,\vec{\mathbf{a}}+\sin\omega t\,\vec{\mathbf{b}}$

$$\frac{d\vec{\mathbf{r}}}{dt}=\frac{d}{dt}(\cos\omega t)\,\vec{\mathbf{a}}+\frac{d}{dt}(\sin\omega t)\,\vec{\mathbf{b}}=-\omega\sin\omega t\,\vec{\mathbf{a}}+\omega\cos\omega t\,\vec{\mathbf{b}}$$

$$\left[\because \frac{d\vec{\mathbf{a}}}{dt}=0,\ \frac{d\vec{\mathbf{b}}}{dt}\right.$$

पुनः t के सापेक्ष अवकलन से,

$$\frac{d^2\vec{\mathbf{r}}}{dt^2}=-\omega^2\cos\omega t\,\vec{\mathbf{a}}-\omega^2\sin\omega t\,\vec{\mathbf{b}}=-\omega^2\,(\cos\omega t\,\vec{\mathbf{a}}+\sin\omega t\,\vec{\mathbf{b}})=-\omega^2\,\vec{\mathbf{r}}$$

$$\therefore\quad \frac{d^2\vec{\mathbf{r}}}{dt^2}+\omega^2\,\vec{\mathbf{r}}=\vec{\mathbf{0}}.$$

(ii) $\vec{\mathbf{r}}\times\frac{d\vec{\mathbf{r}}}{dt}=(\cos\omega t\,\vec{\mathbf{a}}+\sin\omega t\,\vec{\mathbf{b}})\times(-\omega\sin\omega t\,\vec{\mathbf{a}}+\omega\cos\omega t\,\vec{\mathbf{b}})$ [समी० (1) तथा (2)

$$=\omega\cos^2\omega t\,\vec{\mathbf{a}}\times\vec{\mathbf{b}}-\omega\sin^2\omega t\,\vec{\mathbf{b}}\times\vec{\mathbf{a}}\qquad[\because \vec{\mathbf{a}}\times\vec{\mathbf{a}}=\vec{\mathbf{0}},\ \vec{\mathbf{b}}\times\vec{\mathbf{b}}=\vec{\mathbf{0}}]$$

$$=\omega\cos^2\omega t\,\vec{\mathbf{a}}\times\vec{\mathbf{b}}+\omega\sin^2\omega t\,\vec{\mathbf{a}}\times\vec{\mathbf{b}}=\omega\,(\cos^2\omega t+\sin^2\omega t)\,\vec{\mathbf{a}}\times\vec{\mathbf{b}}$$

$$=\omega\,(\vec{\mathbf{a}}\times\vec{\mathbf{b}})\qquad[\because \vec{\mathbf{a}}\times\vec{\mathbf{b}}=-(\vec{\mathbf{b}}\times\vec{\mathbf{a}})]$$

2. अतः t के सापेक्ष अवकलन से

$$\frac{d\vec{\mathbf{r}}}{dt}=\left(\frac{2t}{1+t^2}\right)\vec{\mathbf{i}}+\cos t\,\vec{\mathbf{j}}-2t\,\vec{\mathbf{k}}\qquad\left[\because \frac{d}{dt}\log t=\frac{1}{t}\right]\quad\ldots$$

$$\therefore\quad t=0\text{ पर }\left(\frac{d\vec{\mathbf{r}}}{dt}\right)_{t=0}=0\,\vec{\mathbf{i}}+\cos 0\,\vec{\mathbf{j}}-2.0.\vec{\mathbf{k}}\qquad[\because \cos 0=1]$$

$$=\vec{\mathbf{j}}$$

3. प्रश्न से $\vec{\mathbf{r}}=\vec{\mathbf{a}}\,e^{nt}+\vec{\mathbf{b}}\,e^{-nt}$ ...(

$$\therefore\quad \frac{d\vec{\mathbf{r}}}{dt}=\vec{\mathbf{a}}\,n\,e^{nt}-\vec{\mathbf{b}}\,n\,e^{-nt}\qquad[t\text{ के सापेक्ष अवकलन से}]$$

$$\Rightarrow\quad \frac{d^2\vec{\mathbf{r}}}{dt^2}=\vec{\mathbf{a}}\,n^2\,e^{nt}+\vec{\mathbf{b}}\,n^2\,e^{-nt}\qquad[\text{पुनः } t\text{ के सापेक्ष अवकलन से}]$$

$$\Rightarrow\quad \frac{d^2\vec{\mathbf{r}}}{dt^2}=n^2\,(\vec{\mathbf{a}}\,e^{nt}+\vec{\mathbf{b}}\,e^{-nt})$$

$\Rightarrow \quad \dfrac{d^2 \vec{\mathbf{r}}}{dt^2} = n^2 \vec{\mathbf{r}}$ [(1) से] $\quad \therefore \dfrac{d^2 \vec{\mathbf{r}}}{dt^2} - n^2 \vec{\mathbf{r}} = 0$ सिद्ध हुआ।

4. यहाँ $\dfrac{d}{dt}(\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = \dfrac{d\vec{\mathbf{u}}}{dt} \times \vec{\mathbf{v}} + \vec{\mathbf{u}} \times \dfrac{d\vec{\mathbf{v}}}{dt} = (\vec{\mathbf{w}} \times \vec{\mathbf{u}}) \times \vec{\mathbf{v}} + \vec{\mathbf{u}} \times (\vec{\mathbf{w}} \times \vec{\mathbf{v}})$

[$\dfrac{d\vec{\mathbf{u}}}{dt}$ तथा $\dfrac{d\vec{\mathbf{v}}}{dt}$ के मान रखने पर]

$= (\vec{\mathbf{v}} \cdot \vec{\mathbf{w}})\vec{\mathbf{u}} - (\vec{\mathbf{v}} \cdot \vec{\mathbf{u}})\vec{\mathbf{w}} + (\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})\vec{\mathbf{w}} - (\vec{\mathbf{u}} \cdot \vec{\mathbf{w}})\vec{\mathbf{v}}$ [सदिश गुणन की परिभाषा से]

$= (\vec{\mathbf{v}} \cdot \vec{\mathbf{w}})\vec{\mathbf{u}} - (\vec{\mathbf{u}} \cdot \vec{\mathbf{w}})\vec{\mathbf{v}}$ $[\because \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{v}} \cdot \vec{\mathbf{u}}]$

$= (\vec{\mathbf{w}} \cdot \vec{\mathbf{v}})\vec{\mathbf{u}} - (\vec{\mathbf{w}} \cdot \vec{\mathbf{u}})\vec{\mathbf{v}} = \vec{\mathbf{w}} \times (\vec{\mathbf{u}} \times \vec{\mathbf{v}})$ [सदिश गुणन की परिभाषा से]

$\therefore \quad \dfrac{d}{dt}(\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = \vec{\mathbf{w}} \times (\vec{\mathbf{u}} \times \vec{\mathbf{v}})$ सिद्ध हुआ।

5. $\dfrac{d}{d\theta}\{\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}})\} = \dfrac{d\vec{\mathbf{a}}}{d\theta} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) + \vec{\mathbf{a}} \times \dfrac{d}{d\theta}(\vec{\mathbf{b}} \times \vec{\mathbf{c}})$

$= \dfrac{d\vec{\mathbf{a}}}{d\theta} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) + \vec{\mathbf{a}} \times \left\{\dfrac{d\vec{\mathbf{b}}}{d\theta} \times \vec{\mathbf{c}} + \vec{\mathbf{b}} \times \dfrac{d\vec{\mathbf{c}}}{d\theta}\right\}$

$= (\cos\theta\,\vec{\mathbf{i}} - \sin\theta\,\vec{\mathbf{j}} + \vec{\mathbf{k}}) \times \{(\cos\theta\,\vec{\mathbf{i}} - \sin\theta\,\vec{\mathbf{j}} - 3\vec{\mathbf{k}}) \times (2\vec{\mathbf{i}} + 3\vec{\mathbf{j}} - \vec{\mathbf{k}})$

$+ (\sin\theta\,\vec{\mathbf{i}} + \cos\theta\,\vec{\mathbf{j}} + \theta\vec{\mathbf{k}}) \times \{(-\sin\theta\,\vec{\mathbf{i}} - \cos\theta\,\vec{\mathbf{j}}) \times (2\vec{\mathbf{i}} + 3\vec{\mathbf{j}} - \vec{\mathbf{k}})\}$

$+ (\cos\theta\,\vec{\mathbf{i}} - \sin\theta\,\vec{\mathbf{j}} - 3\vec{\mathbf{k}}) \times 0\}$

[$\vec{\mathbf{a}}, \vec{\mathbf{b}}$ तथा $\vec{\mathbf{c}}$ के मान रखकर अवकलन से]

अब $\theta = 0$ पर, $\cos\theta = 1, \sin\theta = 0$

$\therefore \quad \dfrac{d}{d\theta}\{\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}})\} = (\vec{\mathbf{i}} + \vec{\mathbf{k}}) \times \{(\vec{\mathbf{i}} - 3\vec{\mathbf{k}}) \times (2\vec{\mathbf{i}} + 3\vec{\mathbf{j}} - \vec{\mathbf{k}})\}$

$+ (\vec{\mathbf{j}}) \times \{-\vec{\mathbf{j}} \times (2\vec{\mathbf{i}} + 3\vec{\mathbf{j}} - \vec{\mathbf{k}}) + (\vec{\mathbf{i}} - 3\vec{\mathbf{k}}) \times 0\}$

या $\quad \dfrac{d}{d\theta}\{\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}})\} = (\vec{\mathbf{i}} + \vec{\mathbf{k}}) \times (9\vec{\mathbf{i}} - 5\vec{\mathbf{j}} + 3\vec{\mathbf{k}}\} + \vec{\mathbf{j}} \times \{\vec{\mathbf{i}} + 2\vec{\mathbf{k}}\}$

$= 9 \times 0 - 5\vec{\mathbf{k}} - 3\vec{\mathbf{j}} + 9\vec{\mathbf{j}} + 2\vec{\mathbf{i}} - \vec{\mathbf{k}} + 5\vec{\mathbf{i}} + 0 - \vec{\mathbf{k}} + 2\vec{\mathbf{i}}$

$\left[\because \vec{\mathbf{i}} \times \vec{\mathbf{i}} = \vec{\mathbf{j}} \times \vec{\mathbf{j}} = \vec{\mathbf{k}} \times \vec{\mathbf{k}} = 0 \right.$
$\left. \vec{\mathbf{i}} \times \vec{\mathbf{j}} = \vec{\mathbf{k}} \text{ etc.} \right]$

$= 5\vec{\mathbf{i}} + 6\vec{\mathbf{j}} - 5\vec{\mathbf{k}} + 2\vec{\mathbf{i}} - \vec{\mathbf{k}} = 7\vec{\mathbf{i}} + 6\vec{\mathbf{j}} - 6\vec{\mathbf{k}}$

6. दिया गया है $\vec{\mathbf{r}} = \cos nt\,\hat{\mathbf{i}} + \sin nt\,\hat{\mathbf{j}}$ ...(1)

समी० (1) का t के सापेक्ष अवकलन से

$\dfrac{d\vec{\mathbf{r}}}{dt} = -n\sin nt\hat{\mathbf{j}} + n\cos nt\hat{\mathbf{j}}$ ...(2)

$$\therefore \quad \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{F}}}{dt} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos nt & \sin nt & 0 \\ -n\sin nt & n\cos nt & 0 \end{vmatrix}$$

$$= \hat{\mathbf{k}}\,[n\cos^2 nt + r\sin^2 nt]$$

$$= \hat{\mathbf{k}}\,.\,n\,[\cos^2 nt + \sin^2 nt] = k\hat{\mathbf{n}}$$

(2) का t सापेक्ष अवकलन से

$$\frac{d^2\vec{\mathbf{r}}}{dt^2} = -n^2\cos nt\hat{\mathbf{i}} - n^2\sin nt\hat{\mathbf{j}} = -n^2\,(\cos nt\hat{\mathbf{i}} + \sin nt\hat{\mathbf{j}}) = -n^2\,\vec{\mathbf{r}}$$

i.e., $$\frac{d^2\vec{\mathbf{r}}}{dt^2} = -n^2\,\vec{\mathbf{r}}$$

7. यहाँ $\vec{\mathbf{r}} = 5t^2\,\vec{\mathbf{i}} + t\,\vec{\mathbf{j}} - 3t\,\vec{\mathbf{k}}$ जिससे $\frac{d\vec{\mathbf{r}}}{dt} = 10t\,\vec{\mathbf{i}} + \vec{\mathbf{j}} - 3t^2\,\vec{\mathbf{k}}$

$$\therefore \quad \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{r}}}{dt} = [(5t^2\,\vec{\mathbf{i}} + t\,\vec{\mathbf{j}} - t^3\,\vec{\mathbf{k}}) \times (10t\,\vec{\mathbf{i}} + \vec{\mathbf{j}} - 3t^2\,\vec{\mathbf{k}}]$$

$$5t^2\,\vec{\mathbf{k}} + 15t^4\,\vec{\mathbf{j}} - 10t^2\,\vec{\mathbf{k}} - 3t^3\,\vec{\mathbf{i}} - 10t^4\,\vec{\mathbf{j}} + t^3\,\vec{\mathbf{i}}$$

$$= -2t^3\,\vec{\mathbf{i}} + 5t^4\,\vec{\mathbf{j}} - 5t^2\,\vec{\mathbf{k}} \qquad \left[\because \vec{\mathbf{i}} \times \vec{\mathbf{j}} = \vec{\mathbf{k}},\ \vec{\mathbf{j}} \times \vec{\mathbf{k}} = \vec{\mathbf{i}},\ \vec{\mathbf{k}} \times \vec{\mathbf{i}} = \vec{\mathbf{j}} \atop \vec{\mathbf{i}} \times \vec{\mathbf{i}} = \vec{\mathbf{j}} \times \vec{\mathbf{j}} = \vec{\mathbf{k}} = \vec{\mathbf{k}} \times \vec{\mathbf{k}} = 0\right]$$

$$\therefore \quad \int_1^2 \vec{\mathbf{r}} \times \frac{d^2\vec{\mathbf{r}}}{dt^2}\,dt = \left[\vec{\mathbf{r}} \times \frac{d^2\vec{\mathbf{r}}}{dt}\right]_1^2 = [-2t^3\,\vec{\mathbf{i}} + 5t^4\,\vec{\mathbf{j}} - 5t^2\,\vec{\mathbf{k}}]_1^2$$

$$= (-16\,\vec{\mathbf{i}} + 80\,\vec{\mathbf{j}} - 20\,\vec{\mathbf{k}} + 2\,\vec{\mathbf{i}} - 5\,\vec{\mathbf{j}} + 5\,\vec{\mathbf{k}})$$

$$= -14\,\vec{\mathbf{i}} + 75\,\vec{\mathbf{j}} - 15\,\vec{\mathbf{k}}$$

8. $$\int_0^1 (e^{3t}\,\vec{\mathbf{i}} + e^{-2t}\,\vec{\mathbf{j}} + e^{4t}\,\vec{\mathbf{k}})\,dt = \left[\frac{e^{3t}}{3}\,\vec{\mathbf{i}} + \frac{e^{-2t}}{-2}\,\vec{\mathbf{j}} + \frac{e^{4t}}{4}\,\vec{\mathbf{k}}\right]_0^1$$

$$= \left[\left(\frac{e^3}{3}\,\vec{\mathbf{i}} - \frac{e^{-2}}{2}\,\vec{\mathbf{j}} + \frac{e^4}{4}\,\vec{\mathbf{k}}\right) - \left(\frac{1}{3}\,\vec{\mathbf{i}} - \frac{1}{2}\,\vec{\mathbf{j}} + \frac{1}{4}\,\vec{\mathbf{k}}\right)\right]$$

$$= \frac{1}{3}(e^3 - 1)\,\vec{\mathbf{i}} - \frac{1}{2}(e^{-2} - 1)\,\vec{\mathbf{j}} + \frac{1}{4}(e^4 - 1)\,\vec{\mathbf{k}}$$

9. (i) $\vec{\mathbf{r}}\,.\,\vec{\mathbf{s}} = \{t\,\vec{\mathbf{i}} - t^2\,\vec{\mathbf{j}} + (t-1)\,\vec{\mathbf{k}}\}\,.\,\{2t^2\,\vec{\mathbf{i}} + 6t\,\vec{\mathbf{k}}\}$

$$= 2t^3 + 6\,(t-1)\,t \qquad \left[\because \vec{\mathbf{i}}\,.\,\vec{\mathbf{j}} = \vec{\mathbf{j}}\,.\,\vec{\mathbf{k}} = 0 \atop \vec{\mathbf{i}}\,.\,\vec{\mathbf{i}} = \vec{\mathbf{k}}\,.\,\vec{\mathbf{k}} = 1\right]$$

$$= 2t^3 + 6t^2 - 6t$$

$$\therefore \quad \int_0^1 \vec{\mathbf{r}}\,.\,\vec{\mathbf{s}}\,dt = \int_0^1 (2t^3 + 6t^2 - 6t)\,dt = \left[\frac{2t^4}{4} + \frac{6t^3}{3} - \frac{6t^2}{2}\right]_0^1 = \left[\frac{t^4}{2} + 2t^3 - 3t^2\right]_0^1$$

$$= \frac{1}{2}(1)^4 + 2(1)^3 - 3(1)^2 = \frac{1}{2} + 2 - 3 = -\frac{1}{2}$$ उत्तर

(ii) $\because$ $$\vec{\mathbf{r}} \times \vec{\mathbf{s}} = \left\{t\,\vec{\mathbf{i}} - t^2\,\vec{\mathbf{j}} + (t-1)\,\vec{\mathbf{k}}\right\} \times (2t^2\,\vec{\mathbf{i}} + 6t\,\vec{\mathbf{k}}) = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ t & -t^2 & t-1 \\ 2t^2 & 0 & 6t \end{vmatrix}$$

$$= \vec{\mathbf{i}}\,(-6t^3 - 0) - (6t^2 - 2t^3 + 2t^2)\,\vec{\mathbf{j}} + \vec{\mathbf{k}}\,(0 + 2t^4)$$

$$= -6t^3\,\vec{\mathbf{i}} + (2t^3 - 8t^2)\,\vec{\mathbf{j}} + 2t^4\,\vec{\mathbf{k}}$$

$$\therefore \int_0^2 (\vec{\mathbf{r}} \times \vec{\mathbf{s}})\,dt = \int_0^2 \left\{-6t^3\,\vec{\mathbf{i}} + (2t^3 - 8t^2)\,\vec{\mathbf{j}} + 2t^4\,\vec{\mathbf{k}}\right\} dt$$

$$= -6\,\vec{\mathbf{i}}\left[\frac{t^4}{4}\right]_0^2 + 2\,\vec{\mathbf{j}}\left[\frac{t^4}{4} - \frac{4t^3}{3}\right]_0^2 + \frac{2}{5}\vec{\mathbf{k}}\,[t^5]_0^2$$

$$= -6\,\vec{\mathbf{i}}\left[\frac{2^4}{4}\right] + 2\,\vec{\mathbf{j}}\left[\frac{2^4}{4} - 4 \times \frac{2^3}{3}\right] + \frac{2}{5}\vec{\mathbf{k}}\,[2^5]$$

$$= -6\,\vec{\mathbf{i}} \times 4 + 2\,\vec{\mathbf{j}}\left(4 - \frac{32}{3}\right) + 2\vec{\mathbf{k}}\left(\frac{32}{5}\right) = -24\,\vec{\mathbf{i}} - \frac{40}{3}\vec{\mathbf{j}} + \frac{64}{5}\vec{\mathbf{k}}$$ उत्तर

10. यहाँ $\vec{\mathbf{a}} \times \dfrac{d^2\vec{\mathbf{r}}}{dt^2} = \vec{\mathbf{b}}$ $\Rightarrow$ $\displaystyle\int\left(\vec{\mathbf{a}} \times \frac{d^2\vec{\mathbf{r}}}{dt^2}\right)dt = \int \vec{\mathbf{b}}\,dt$ $\Rightarrow$ $\vec{\mathbf{a}} \times \dfrac{d\vec{\mathbf{r}}}{dt} = \vec{\mathbf{b}}\,t + \vec{\mathbf{c}}$

$\Rightarrow$ $\displaystyle\int \vec{\mathbf{a}} \times \frac{d\vec{\mathbf{r}}}{dt}\,dt = \int (\vec{\mathbf{b}}\,t + \vec{\mathbf{c}})\,dt$ [पुन: समाकलित करने पर]

$\Rightarrow$ $\vec{\mathbf{a}} \times \vec{\mathbf{r}} = \vec{\mathbf{b}}\,\dfrac{t^2}{2} + \vec{\mathbf{c}}\,t + \vec{\mathbf{d}}$; जहाँ $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}, \vec{\mathbf{d}}$ अचर सदिश हैं।

11. $$\int_0^3 \int_0^2 \left[\int_0^1 (x + y + z)\,dz\right] dx\,dy$$

$$= \int_0^3 \int_0^2 \left[xz + yz + \frac{z^2}{2}\right]_0^1 dx\,dy$$ [x तथा y को अचर मानने पर]

$$= \int_0^3 \int_0^2 \left[x \times 1 + y \times 1 + \frac{1}{2}\right] dx\,dy$$

$$= \int_0^3 \left[\int_0^2 \left(x + y + \frac{1}{2}\right) dx\right] dy$$

$$= \int_0^3 \left[\frac{x^2}{2} + xy + \frac{x}{2}\right]_0^2 dy$$ [y को अचर मानने पर]

$$=\int_0^3\left[\frac{2^2}{2}+2y+\frac{2}{2}\right]dy=\int_0^3[2+2y+1]\,dy$$

$$=\int_0^3(3+2y)\,dy$$

$$=\left[3y+\frac{2y^2}{2}\right]_0^3=\left[\left(3\times 3+2\times\frac{3^2}{2}\right)-0\right]$$

$$=(9+9)=18$$

उ

12. $$\int_1^2\int_0^{y/2}y\,dy\,dx=\int_1^2\left[\int_0^{y/2}dx\right]y\,dy$$

$$=\int_1^2[x]_0^{y/2}y\,dy$$ [y को अचर मानकर x के सापेक्ष समाकलन से]

$$=\int_1^2\left(\frac{y}{2}-0\right)y\,dy=\frac{1}{2}\int_1^2 y^2dy=\frac{1}{2}\left[\frac{y^3}{3}\right]_1^2$$

$$=\frac{1}{6}[2^3-1^3]=\frac{1}{6}[8-1]=\frac{7}{6}$$

उत्त

13. $$\int_0^b\int_0^a(x^2+y^2)\,dx\,dy=\int_0^b\left[\int_0^a(x^2+y^2)\,dx\right]dy$$

$$=\int_0^b\left[\frac{x^3}{3}+y^2x\right]_0^a dy$$ [y को अचर मानकर समाकलन करने पर]

$$=\int_0^b\left[\left(\frac{a^3}{3}+ay^2\right)-0\right]dy=\int_0^b\left(\frac{a^3}{3}+ay^2\right)dy=\frac{a^3}{3}\int_0^b dy+a\int_0^b y^2dy$$

$$=\frac{a^3}{3}[y]_0^b+a\left[\frac{y^3}{3}\right]_0^b=\frac{a^3}{3}(b-0)+\frac{a}{3}(b^3-0)$$

अर्थात् $$\int_0^b\int_0^a(x^2+y^2)\,dx=\frac{1}{3}\{a^3b+ab^3\}=\frac{1}{3}ab\,(a^2+b^2)$$

उत्त

14. $$\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz=\int_0^1\int_0^1\left[\int_0^1 e^{x+y+z}\,dx\right]dy\,dz$$

$$=\int_0^1\int_0^1[e^{x+y+z}]_0^1\,dy\,dz=\int_0^1\int_0^1[e^{1+y+z}-e^{0+y+z}]\,dy\,dz$$

$$=\int_0^1\left[\int_0^1[e^{1+y+z}]_0^1\,dy\right]dz=\int_0^1[(e^{1+1+z}-e^{1+z})-e^{1+0+z}-e^{0+z})]\,dz$$

$$=\int_0^1[e^{2+z}-2e^{1+z}+e^z]\,dz=e^{2+z}-2e^{1+z}+e^z]_0^1$$

$$=(e^{2+1}-2e^{1+1}+e^1)-(e^{2+0}-2e^{1+0}+e^0)]$$

$$=[e^3-2e^2-e-e^2+2e+1)]=e^3-3e^2+3e+1=(e-1)^3$$

Ans.

□□□